

WIS/2/05-JAN-DPP

hep-th/0501012

A new Ricci flat geometry

Shesansu Sekhar Pal

*Weizmann institute of science,
76100 Rehovot, Israel
shesansu.pal@weizmann.ac.il*

Abstract

We are proposing a new Ricci flat metric constructed from an infinite family of Sasaki-Einstein, $Y^{(p,q)}$, geometries. This geometry contains a free parameter s and in the $s \rightarrow 0$ limit we get back the usual CY. When this geometry is probed both by a stack of D3 and fractional D3 branes then the corresponding supergravity solution is found which is a warped product of this new 6-dimensional geometry and the flat $R^{3,1}$. This solution in the specific limit as mentioned above reproduces the solution found in hep-th/0412193. The integrated five-form field strength over $S^2 \times S^3$ goes logarithmically but the argument of Log function is different than has been found before.

1 Introduction

The study of AdS/CFT [1]¹ has been instrumental in sharpening our understanding of various gravitational theories with singularity, especially branes probing various conical singularities. The famous example before [8] is the conifold where probing of the tip of conifold with various D3-branes provides us the $AdS_5 \times T^{1,1}$ space which in the corresponding dual field theory [3] would become conformal or nonconformal depending on the absence or presence of fractional D3-branes. The latter are D5 branes wrapped on the 2-cycles of the conifold. More interestingly, these field theories preserve four supercharges in four spacetime dimensions i.e. it preserves $\mathcal{N} = 1$ supersymmetry.

There has been an obstacle to compute the scaling dimensions or the anomalous dimensions of the chiral superfields that appear in these supersymmetric field theories due to the appearance of various R-symmetries. In [4], it has been suggested to consider the R-currents which maximizes the conformal anomaly, a , over all the possible R-currents which appear in the superconformal multiplet. There has been interesting developments on the a -maximization principle in [5, 6].

Recently, in [8] an infinite family of Sasaki-Einstein manifold $Y^{(p,q)}$ has been constructed by studying supersymmetric AdS solutions in 11-dimensional supergravity [7]. These geometries are described by two positive integers p and q with a restriction $q < p$. The topology of this space is $S^2 \times S^3$ and have the isometry group $SU(2) \times U(1)^2$. It has been shown in [8, 9] that there exists a Killing vector called Reeb vector which in the dual field theory is isomorphic to the R-symmetry, which is $\sim \frac{\partial}{\partial \psi}$ and depending on the orbits of this Killing vector field one gets regular, quasi-regular and irregular SE manifolds. This newly found infinite family of Sasaki-Einstein geometries falls into the last class i.e. of irregular type. One of the intrinsic property of the dual field theory associated to this kind irregular geometry is that their central charges are irrational and hence the volumes of these geometries are irrational too. As we have mentioned already in the previous paragraph the dual field theory is an $\mathcal{N} = 1$ superconformal field theory when there is no fractional D3 branes otherwise it will be non-conformal field theory like [13] with the same amount of supersymmetry. These conformal field theories has been studied in [9, 10, 11] and has been shown to possess Seiberg duality [14].

In a related development in [12] found the supergravity solution by probing these singular Calabi-Yau's with a stack of both D3 and fractional D3 branes. These solution indeed shows the characteristic feature of Seiberg duality i.e. the integrated five form field strength over $S^2 \times S^3$ goes as $\log r$.

In this paper we are proposing a new 6-dimensional Ricci-flat geometry unlike the usual— a cone over $Y^{(p,q)}$ i.e. eq.(7). This new Ricci flat geometry contains a free parameter s and in the limit of $s \rightarrow 0$ it gives back the eq. (7). We also present the supergravity solution for these Ricci flat geometry and in the above mentioned limit we get back the solution presented in [12]. An inter-

¹For a review [2].

esting point to note that the warp factor depends on both the radial coordinate and the angular coordinate y , which has been mentioned in [12]. But, for our choice of metric ansatz eq.(10), it follows that the y dependent part of h is universal i.e its form is same irrespective of the form of G_1 and G_2 , the $H(y)$ of eq.(19).

The appearance of the parameter s in the geometry corresponds to a closed string moduli and it would be interesting to understand whether this moduli corresponds to a normalisable or non normalisable mode,² and the presence or absence of singularity at $\rho = 0$, which we will not do that here but will do that in our future studies.

2 The new geometry

The new six dimensional Ricci flat geometry that we are proposing is guessed from the existing 5-dimensional $Y^{(p,q)}$ Sasaki-Einstein metric. Its form look like

$$ds_6^2 = K^{-1}(\rho)d\rho^2 + K(\rho)\rho^2 e^{\psi^2} + (\rho^2 + s^2)(e^{\theta^2} + e^{\phi^2} + e^{y^2} + e^{\beta^2}), \quad (1)$$

where the one forms are defined as

$$\begin{aligned} e^\theta &= \sqrt{\frac{1-cy}{6}}d\theta, & e^\phi &= \sqrt{\frac{1-cy}{6}}\sin\theta d\phi, \\ e^y &= \frac{1}{\sqrt{w(y)v(y)}}dy, & e^\beta &= \frac{\sqrt{w(y)v(y)}}{6}(d\beta + c \cos\theta d\phi), \\ e^\psi &= \frac{1}{3}[d\psi - \cos\theta d\phi + y(d\beta + c \cos\theta d\phi)] \end{aligned} \quad (2)$$

and ³

$$\begin{aligned} K(\rho) &= \frac{\rho^4 + 3s^2\rho^2 + 3s^4}{(\rho^2 + s^2)^2}, \\ w(y) &= \frac{2(b-y^2)}{(1-cy)}, & v(y) &= \frac{b-3y^2+2cy^3}{b-y^2} \end{aligned} \quad (3)$$

As discussed in [8] that one can rescale the coordinate y and the parameter c can be set to unity i.e. $c = 1$. From now onwards we shall follow this except towards the end of this section. The above solution makes sense only when the y coordinate stays between the two smallest roots of $b - 3y^2 + 2y^3$ and are given by

$$\begin{aligned} y_1 &= \frac{1}{4p} \left(2p - 3q - \sqrt{4p^2 - 3q^2} \right), \\ y_2 &= \frac{1}{4p} \left(2p + 3q - \sqrt{4p^2 - 3q^2} \right), \end{aligned} \quad (4)$$

²I would like to thank M. Berkooz for suggesting to study this.

³We have written $v(y)$ as opposed to $q(y)$, following [12], may be to avoid the confusion of q appearing in $Y^{(p,q)}$ and in the geometry.

where

$$b = \frac{1}{2} - \frac{p^2 - 3q^2}{4p^3} \sqrt{4p^2 - 3q^2}. \quad (5)$$

Following [8], one can define $\alpha = -\beta/6 - \psi/6$ and re express the metric written above in a different form and for our purpose the exact form that is not important. The aim of introducing the coordinate α is to ⁴ mention that this coordinate has a period of $2\pi\ell$. So, the ranges of various coordinates are: $0 \leq \theta \leq \pi, 0 \leq \phi \leq 2\pi, 0 \leq \psi \leq 2\pi, y_1 \leq y \leq y_2$ and $0 \leq \alpha \leq 2\pi\ell$.

The space $Y^{(p,q)}$ has two independent parameters, corresponding to the two Chern numbers and the period of α and the volume of this space depends on these two parameters (p, q) .

$$\ell = \frac{q}{3q^2 - 2p^2 + p\sqrt{4p^2 - 3q^2}}$$

$$\text{Vol}[Y^{(p,q)}] = \frac{q^2[2p + \sqrt{4p^2 - 3q^2}]}{3p^2[3q^2 - 2p^2 + p\sqrt{4p^2 - 3q^2}]}, \quad (6)$$

with a restriction of $p > q$.

It is easy to note that in the $s \rightarrow 0$ limit we do get back the usual way of constructing Calabi-Yau from the 5-dimensional $Y^{(p,q)}$ geometry i.e.

$$ds^2 = dr^2 + r^2 ds_{Y^{(p,q)}}^2. \quad (7)$$

Let us take the form of $K(\rho)$ that appear in eq.(1) as

$$K(\rho) = \frac{c_1 + \rho^6 + 3\rho^4 s^2 + 3s^4 \rho^2}{\rho^2(\rho^2 + s^2)^2}, \quad (8)$$

where c_1 is a constant⁵. This metric is also Ricci-flat in fact it is related to the earlier $K(\rho)$ by some change of coordinates and defining new constants. The interesting thing to note that in the limit of setting $c_1 = s^6$, we do get back our singular geometry i.e. eq.(7). The most interesting point is that this geometry is a Calabi-Yau. Its Kahlerian behavior is shown in [16].

Let us recall the geometry of the resolved conifold metric from [17]

$$ds^2 = \tilde{K}^{-1}(r) dr^2 + \tilde{K}(r) r^2 e^{\psi^2} + (r^2 + s_1^2)(e^{\theta_1^2} + e^{\phi_1^2}) + (r^2 + s_2^2)(e^{\theta_2^2} + e^{\phi_2^2}), \quad (9)$$

where $\tilde{K}(r) = \frac{c_2 + r^6 + 3/2(s_1^2 + s_2^2)r^2 + 3r^2 s_1^2 s_2^2}{r^2(r^2 + s_1^2)(r^2 + s_2^2)}$ with $c_2 = \frac{s_4}{2}(3s_2^2 - s_1^2)$.

It is easy to see that in the $c = 0$ limit of eq.(1) with eq.(8) gives the same metric as eq.(9) with $s_1 = s_2$ and ignoring the exact form of c_2 . From this it is tempting to think that eq.(1) with eq.(8) may give hints to the geometry of resolved ⁶ $Y^{p,q}$ geometry.

⁴In [8], they first wrote down their metric in $(\theta, \phi, y, \beta, \alpha)$ coordinates.

⁵We shall work in the $c_1 = 0$ limit in next section.

⁶More on it latter.

3 The solution

We shall find the solution for the geometry written in eq.(1) by first constructing a closed 3-form field strength which obeys the imaginary-self-duality (ISD) condition with respect to the geometry given in eq.(1). We believe that this is enough to show that the supergravity solution preserve $\mathcal{N} = 1$ supersymmetry. In [15], it has been shown whenever the 3-form field strength becomes (2,1) and obeys the ISD condition and more importantly, for the case when the warp factor $h(\tau)$ is a function only of the radial coordinate τ then it necessarily preserves supersymmetry. However, for the irregular SE geometries the warp factor is a function of both the radial and one of the angular coordinate, y . So, it is important to check that this solution preserves supersymmetry ⁷. For the form of our 6-dimensional metric we shall get the solution with the following ansatz to metric.

The ansatz to the 10-dimensional geometry is

$$ds^2 = h^{-1/2} ds_4^2 + h^{1/2} \left(G_1^2(\tau) [d\tau^2 + e^{\psi^2}] + G_2^2(\tau) [e^{\theta^2} + e^{\phi^2} + e^{y^2} + e^{\beta^2}] \right). \quad (10)$$

For $G_1 = G_2 = r$ and $\tau = \ln r$, we get back the 6-dimensional geometry eq.(7) and the solution derived in [12]. Whereas in our case

$$G_1 = \sqrt{K(\rho)}\rho, \quad G_2 = \sqrt{\rho^2 + s^2}, \quad \tau = \frac{1}{6} \ln[\rho^6 + 3s^2\rho^4 + 3s^4\rho^2]. \quad (11)$$

The dilaton, ϕ is ⁸, assumed to be constant and the axion, C , is set to zero i.e.

$$e^\phi \equiv e^{\phi_0} = g_s, \quad C = 0. \quad (12)$$

The NS-NS 2-form potential, B_2 , has the form

$$B_2 = g_s M K K' f(\tau) F(y) [e^\theta \wedge e^\phi - e^y \wedge e^\beta] \quad (13)$$

with the form of $F(y)$ as

$$F(y) = \frac{1}{(1-y)^2}. \quad (14)$$

The corresponding 3-form field strength, H_3 and RR 3-form field strength, F_3 is

$$\begin{aligned} H_3 &= g_s M K K' F(y) \frac{df(\tau)}{d\tau} d\tau \wedge (e^\theta \wedge e^\phi - e^y \wedge e^\beta), \\ F_3 &= -M K K' F(y) \frac{df(\tau)}{d\tau} e^\psi \wedge (e^\theta \wedge e^\phi - e^y \wedge e^\beta). \end{aligned} \quad (15)$$

In order to be consistent with the Bianchi identity of these 3-form field strengths we shall set $\frac{df(\tau)}{d\tau} = 1$ and the quantization condition of F_3 imply [12]

$$K' = 4\pi^2 \alpha', \quad K = \frac{9}{8\pi^2} (p^2 - q^2). \quad (16)$$

⁷which we will not do it here.

⁸The dilaton should not be confused with the angular coordinate appear in the geometry.

The self dual five-form field strength is

$$\begin{aligned}\tilde{F}_5 = & -\frac{h^{-2}}{g_s} \frac{\partial h}{\partial \tau} d\tau \wedge dx^0 \wedge \cdots \wedge dx^3 - \frac{h^{-2}}{g_s} \frac{\partial h}{\partial y} dy \wedge dx^0 \wedge \cdots \wedge dx^3 \\ & -\frac{1}{g_s} \frac{\partial h}{\partial \tau} G_2^4 e^\theta \wedge e^\phi \wedge e^y \wedge e^\psi + \frac{1}{g_s} \frac{\partial h}{\partial y} \sqrt{wv} G_1^2 G_2^2 d\tau \wedge e^\theta \wedge e^\phi \wedge e^\beta \wedge e^\psi.\end{aligned}\quad (17)$$

It is easy to note that the warp factor is a function of both the radial coordinate τ and the angular coordinate y , as we have written in the expression of five form field strength, because of the appearance of $F(y)$ in the three form field strengths i.e. if we look at the Bianchi identity or the equation of motion associated to the five form field strength then the RHS of this equation, $H_3 \wedge F_3$ depends on $F(y)$. The explicit form of the warp factor $h(\tau, y)$ is

$$h = 4B \int_\tau \frac{\tau d\tau}{G_2^4(\tau)} - \tilde{B} \int_\tau \frac{d\tau}{G_2^2} + \frac{1}{G_1^2 G_2^2} H(y) + \text{constant}, \quad (18)$$

where \tilde{B} is a constant of integration and $B = \frac{g_s^2 K^2 K'^2}{2(1-y_1)^2(1-y_2)^2}$ and $H(y)$ is

$$H(y) = -\frac{(g_s M K K')^2}{2(b-1)} \left(\frac{1}{1-y} + \frac{(1+2y_1)(1+2y_2)\log(y_3-y)}{2(b-1)} \right) + \text{constants}. \quad (19)$$

It is interesting to note that all the roots of $b - 3y^2 + 2y^3$ appear in $H(y)$ and more importantly, this function do not diverges for $y_1 \leq y \leq y_2$ [12]. Given the 10-dimensional metric ansatz in eq.(10) this form of $H(y)$ is universal in the sense that one will get a term in the warp factor which depends only on the y coordinate and whose form is that in eq.(19) even if the 6-dimensional Ricci flat metric is of the $ds^2 = G_1^2(\tau)[d\tau^2 + e^{\psi^2}] + G_2^2(\tau)[e^{\theta^2} + e^{\phi^2}] + G_3^2(\tau)e^{y^2} + G_4^2(\tau)e^{\beta^2}$ form.

The tilde five form field strength integrated over the “5-cycle= $S^2 \times S^3$ ” is

$$\oint_{S^5} \tilde{F}_5 = \frac{1}{g_s} \left[4B\tau - \text{const.} + 2 \left(\frac{G_2 \frac{dG_2}{d\tau}}{G_1^2} + \frac{G_2^2 \frac{dG_1}{d\tau}}{G_1^3} \right) \right] \oint_{S^5} e^\theta \wedge e^\phi \wedge e^y \wedge e^\beta \wedge e^\psi \quad (20)$$

Let us now evaluate the warp factor $h(\rho, y)$ and $\oint_{S^5} \tilde{F}_5$ for eq.(11) and the results of it in terms of our radial coordinate ρ are

$$h(\rho, y) = \frac{2}{3} B \int_\rho dx \frac{\log[x^6 + 3s^2 x^4 + 3s^4 x^2]}{[x^5 + 3x^3 a^2 + 3s^4 x]} + \frac{H(y)}{(\rho^2 + s^2)^2} + \text{const}, \quad (21)$$

where we have set the constant \tilde{B} to zero. In the $\rho \rightarrow 0$ limit the warp factor has a piece which depends on ρ logarithmically and a piece independent of ρ but depends on the angular coordinate y through $H(y)$ and

$$\oint_{S^5} \tilde{F}_5 = \frac{1}{g_s} \left[\frac{2B}{3} \log[\rho^6 + 3s^2 \rho^4 + 3s^4 \rho^2] - \text{const.} \right]$$

$$+2\left(1 + \frac{(\rho^2 + s^2)(\rho^6 + 3s^2\rho^4 + 3\rho^2s^4 + 3s^6)}{\rho^2(\rho^6 + 4s^2\rho^4 + 6s^4\rho^2 + 3s^6)}\right)\Big]\oint_{S^5} e^\theta \wedge e^\phi \wedge e^y \wedge e^\beta \wedge e^\psi. \quad (22)$$

Note Added: After submitting the paper to arXiv.org, we are informed by C.N. Pope of their paper [16]. For $\Lambda = 0, \lambda = 6, n = 2, \kappa = c_1 - s^6$ and taking the 4-dim geometry as the base of the Einstein-Sasaki metric of $Y^{p,q}$ i.e. $e^{\theta^2} + e^{\phi^2} + e^{y^2} + e^{\beta^2}$ along with a choice for the connection gives rise to our metric.

4 Acknowledgment

We would like to thank O. Aharony, M. Berkooz, Andrei and D. Gepner for many useful discussions and suggestions on the subject also to C. Herzog and C. Pope for useful correspondences. The financial help from Feinberg graduate school is gratefully acknowledged.

References

- [1] J.M. Maldacena, “The large N limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. **2**, 231 (1998) and Int. J. Theor. Phys. **38** 1113 (1999), hep-th/9711200; S.S. Gubser, I.R. Klebanov and A.M. Polyakov, “Gauge theory correlators from non-critical string theory,” Phys. Lett.B **428**, 105 (1998), hep-th/9802109; E. Witten, “Anti-de Sitter space and holography,” Adv. Theor. Math. Phys. **2**, 253 (1998), hep-th/9802150.
- [2] O. Aharony, S.S. Gubser, J.M. Maldacena, H. Ooguri and Y.Oz, “Large N field theories, string theory and gravity,” Phys. Rept. **323**, 183 (2000): hep-th/9905111 and references therein.
- [3] I.R. Klebanov and E. Witten, “Superconformal field theory on three branes at a Calabi-Yau singularity,” Nucl. Phys. B **356**, 199 (1998), hep-th/9807080.
- [4] K. Intriligator and B. Wecht, “The exact superconformal R-symmetry maximizes a,” Nucl. Phys. B. **667**, 183 (2003), hep-th/0304128.
- [5] D. Kutasov, “New results on the ‘a-theorem’ in four dimensional supersymmetric field theory,” hep-th/0312098; D. Kutasov, A. Parnachev and D.A. Sahakyan, “Central charges and $U(1)_R$ symmetries in N=1 super Yang-Mills,” JHEP **0311**, 013 (2003), hep-th/0308071; D. Kutasov and A. Schwimmer, “Lagrange multipliers and couplings in supersymmetric field theory,” Nucl. Phys. B. **702**, 131 (2004), hep-th/0409029.

- [6] E. Barnes, K. Intriligator, B. Wecht and J. Wright, “Evidence for the strongest version of the a-theorem, via a-maximization,” Nucl. Phys. B **702**, 131 (2004), hep-th/0408156.
- [7] J.P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, “Supersymmetric AdS_5 solutions of M-theory,” hep-th/04022153.
- [8] J.P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, “Sasaki-Einstein metrics on $S^2 \times S^3$,” hep-th/0403002; hep-th/0402153.
- [9] D. Martelli and J. Sparks, “Toric geometry, Sasaki-Einstein Manifolds and a New Infinite Class of AdS/CFT Duals,” hep-th/0411238.
- [10] M. Bertolini, F. Bigazzi and A.L. Cotrone, “New checks and subtleties for AdS/CFT and a-maximization,” hep-th/0411249.
- [11] S. Benvenuti, S. Franco, A. Hanany, D. Martelli and J. Sparks, “An infinite family of superconformal quiver gauge theories with Sasaki-Einstein duals,” hep-th/0411264.
- [12] Q.J. Ejaz, C.P. Herzog and I.R. Klebanov, “Cascading RG Flows from New Sasaki-Einstein Manifolds,” hep-th/0412193.
- [13] I.R. Klebanov and M. Strassler, “Supergravity and a confining gauge theory: Duality cascades and χ SB-resolution of naked singularities,” JHEP **0008**, 052 (2000): hep-th/0007191.
- [14] N. Seiberg, “Electric-magnetic duality in supersymmetric nonAbelian gauge theories,” Nucl. Phys. B **435** (1995) 129: hep-th/9411149.
- [15] A. Kehagias, “New type IIB vacua and their F-theory interpretation,” Phys. Lett. B. **435**, 337 (1998), hep-th/9805131; K. Dasgupta, G. Rajesh and S. Sethi, “M-theory, orientifolds and G-flux,” JHEP **9908**, 023 (1999), hep-th/9908088; M. Grana and J. Polchinski, “Supersymmetric three-form flux perturbations on AdS_5 ,” Phys. Rev. D **63**, 026001 (2001), hep-th/0009211; S.S. Gubser, “Supersymmetric and F-theory realization of the deformed conifold with three-form flux,” hep-th/0010010.
- [16] D.N. Page and C.N. Pope, “Inhomogeneous Einstein metrics on complex line bundles,” Class. Quant. Grav. **4**, 213 (1987).
- [17] I.A. Pando Zayas and A. Tseytlin, “3-branes on resolved conifold,” JHEP **0011**, 028 (2000), hep-th/0010088.